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L_1-Induced Static Output Feedback Controller Design and Stability Analysis for Positive Polynomial Fuzzy Systems

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Abstract. The aim of this paper is to study the control synthesis and stability and positivity analysis under L_1-induced performance for positive systems based on a polynomial fuzzy model. In this paper, not only the stability and positivity analysis are studied but also the L_1-induced performance is ensured by designing a static output feedback polynomial fuzzy controller for the positive polynomial fuzzy (PPF) system. In order to improve the flexibility of controller implementation, imperfectly matched premise concept under membership-function-dependent analysis technique is introduced. In addition, although the static output feedback control strategy is more popular when the system states are not completely measurable, a tricky problem that non-convex terms exist in stability and positivity conditions will follow. The nonsingular transformation technique which can transform the non-convex terms into convex ones successfully plays an important role to solve this puzzle. Based on Lyapunov stability theory, the convex positivity and stability conditions in terms of sum of squares (SOS) are obtained, which can guarantee the closed-loop systems to be positive and asymptotically stable under the L_1-induced performance. Finally, in order to test the effectiveness of the derived theory, we show an example in the simulation section.

Keywords: Positive Polynomial Fuzzy-Model-Based (PPFMB) Control Systems, Static Output Feedback Control, Stability Analysis, Sum of Squares (SOS), L_1-induced performance

1 Introduction

Positive systems attract more and more attention from researchers. Due to the unique characteristics of positive systems, for example, the system states always stay in the positive quadrant with the non-negative initial conditions, some scholars begin to investigate this kind of systems from different points of view. In [1], the controller synthesis for positive linear systems with bounded controls was investigated. Considering that the states of positive systems are not obtained completely in some cases, the authors in [2] proposed two iterative algorithms for solving static output feedback (SOF) stabilization problem for LTI multi-input multi-output systems. The work in [3] devotes to the stability of continuous-time positive switched linear systems. Although these works have provided a theoretical foundation for the study of control synthesis and stability analysis of positive systems, these results only work for positive linear systems. However, in practice, many actual systems are complex positive nonlinear systems. Due to the complexity of positive nonlinear systems, many current results for positive linear systems cannot be directly employed. Therefore, it is worth a try to study the control synthesis for positive nonlinear systems.

Designing a state feedback controller for a positive nonlinear system is easy to achieve, but it is more meaningful to design a SOF controller by thinking about the following two aspects: 1) in many cases, the state feedback controller is not available because it is hard to get all of the state information; 2) the implementation cost of state feedback controller is comparatively high. Hence, we focus on investigating SOF controllers for positive nonlinear systems in this paper. Nevertheless, the non-convex terms which are led by system matrix will generally set up a huge barrier for stability and positivity analysis [4]. As a result, how to realize the transformation of non-convex conditions is a challenging problem to be solved. Moreover, as we all know that in order to study different performance of systems, different performance indexes are given, such as, H_∞ control and H_2 control. H_∞ norm is obtained in L_2 signal space, but it cannot express the features of practical positive systems naturally. Relatively speaking, L_1-norm can describe the features of positive systems more accurately, because L_1-norm provides the sum of the values of the components. For example, when the meaning of values is the the number of animals, it is more appropriate to use the L_1-norm than others.

Compared with Takage-Sugeno (T-S) fuzzy model, polynomial fuzzy model has more advantages: Firstly, polynomial fuzzy model allows polynomials in the system matrices and the membership functions (MFs) [5], which means a wider range of nonlinear systems can be expressed by this kind of fuzzy model. Secondly, imperfectly matched premise concept under membership-function-dependent analysis technique is introduced to improve the flexibility of controller implementation. Furthermore, some relaxed methods can be introduced into the stability analysis to reduce the conservativeness [6, 7]. Considering the above two points, we choose the polynomial fuzzy model to express complex positive nonlinear systems. Nevertheless, positive polynomial fuzzy systems are different with general polynomial fuzzy systems because this kind of systems need all subsystem matrices to be Metzler matrices and all the elements of input matrix as well as the output matrix are non-negative [8, 9]. In addition, the positivity of the closed-loop positive polynomial fuzzy control systems also should be ensured, which leads to most of the previous results based on general polynomial fuzzy systems cannot be employed directly. Thereby, the work in this paper is very interesting but also challenging.

The authors in [10] had studied the positivity and stability analysis for a positive polynomial fuzzy system. After a little trial, more efforts were made to design a polynomial fuzzy controller (PFC) based on state feedback control technique in [11]. Considering that it is hard to obtain the full information of state variables in many cases, the authors had a try to design the SOF controllers for positive polynomial fuzzy systems in [4]. However, the L_1-induced performance of positive polynomial fuzzy systems has not yet been taken into account. Noting the importance of robustness that deals with uncertainty exhibiting in practical systems, it is vital to study the L_1-induced SOF control for positive polynomial fuzzy systems. So far, as is known to the authors, there are no results related to this topic, which also gives us a big incentive to do this work.

Within this paper, the PFC is designed based on the SOF control technique. The L_1-induced performance and the stability and positivity analysis of the positivity polynomial fuzzy control systems are carried out. In order to obtain feasible solutions, on the one hand, the non-convex terms in the stability conditions need to be dealt with so that the SOSTOOL [7] can be employed. On the other hand, the convex positive conditions should be ensured so that the positivity of the closed-loop systems can be achieved. Fortunately, nonsingular transformation technique facilitates the work in this paper. The convex SOS-based stability and positivity conditions are derived in terms of Lyapunov stability theory [12]. A simulation example is employed to verify the effectiveness of this approach.

The remainder of the paper consists of the following sections. In Section 2, we mainly introduce some important notations, and show the positive polynomial fuzzy model as well as the PFC based on the SOF control strategy. In Section 3, in terms of the Lyapunov stability theory and L_1-induced performance index, the stability and positivity analysis under the L_1-induced performance of the closed-loop positive polynomial fuzzy control system are derived. In Section 4, we demonstrate the validity of the L_1-induced SOF control scheme for positive polynomial fuzzy systems. In Section 5, a conclusion is given.

2 Preliminaries

2.1 Notation

Throughout this paper, the following notations are employed [13]. The monomial in $\mathbf{x}(t) = [x_1(t), \dots, x_n(t)]^T$ is defined as $x_1^{d_1}(t), \dots, x_n^{d_n}(t)$, where d_k, $k \in \{1, \dots, n\}$, is a non-negative integer. The degree of a monomial is defined as $d = \sum_{k=1}^{n} d_k$. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is shown as finite linear combination of monomials with real coefficients. If a polynomial $\mathbf{p}(\mathbf{x}(t))$ can be expressed as $\mathbf{p}(\mathbf{x}(t)) = \sum_{j=1}^{m} \mathbf{q}_j(\mathbf{x}(t))^2$, where m is a non-zero positive integer and $\mathbf{q}_j(\mathbf{x}(t))$ is a polynomial for all j, it can be concluded that $\mathbf{p}(\mathbf{x}(t)) \geq 0$ is a SOS. For a matrix $\mathbf{N} \in \Re^{m \times n}$ where n_{rs} denotes the element located at the r-th row and s-th column, the expressions $\mathbf{N} \succeq 0$, $\mathbf{N} \succ 0$, $\mathbf{N} \preceq 0$ and $\mathbf{N} \prec 0$ mean that each element n_{rs} is non-negative, positive, non-positive and negative, respectively. $\mathbf{Q}(\mathbf{x}) = \text{diag}(x_1, \ldots, x_n)$ represents $\mathbf{Q}(\mathbf{x})$ is a diagonal matrix with all of the diagonal elements being $x_1, \ldots, x_n$.

2.2 Positive Polynomial Fuzzy Model

A nonlinear positive system is approximated by p polynomial fuzzy rules and the i-th fuzzy rule is shown as follows:

$$\begin{aligned} &\text{Rule } i: \text{IF } f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND} \cdots \text{AND } f_\Psi(\mathbf{x}(t)) \text{ is } M_\Psi^i \\ &\text{THEN } \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t) + \mathbf{B}_{i\omega}\tilde{\mathbf{w}}(t), \\ \mathbf{z}(t) = \mathbf{D}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{E}_i(\mathbf{x}(t))\mathbf{u}(t) + \mathbf{E}_{i\omega}\tilde{\mathbf{w}}(t), \end{cases} \end{aligned} \tag{1}$$

where $f_l(\mathbf{x}(t))$, is the premise variable, Ψ is a positive integer; M_l^i is the fuzzy set of the i-th rule corresponding to the function $f_l(\mathbf{x}(t))$; $\mathbf{x}(t) \in \Re^n$ is the system state vector; $\mathbf{u}(t) \in \Re^m$ is the input vector; $\tilde{\mathbf{w}}(t) \in \Re^p$ is the disturbance signal; $\mathbf{z}(t) \in \Re^q$ is the and controlled output; $\mathbf{A}_i(\mathbf{x}(t))$, $\mathbf{B}_i(\mathbf{x}(t))$, $\mathbf{B}_{i\omega}$, $\mathbf{D}_i(\mathbf{x}(t))$, $\mathbf{E}_i(\mathbf{x}(t))$ and $\mathbf{E}_{i\omega}$ are the system matrices with right dimensions.

The dynamics of the positive polynomial fuzzy system is expressed as follows:

$$\begin{cases} \dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))(\mathbf{A}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t) + \mathbf{B}_{i\omega}\tilde{\mathbf{w}}(t)), \\ \mathbf{z}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))(\mathbf{D}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{E}_i(\mathbf{x}(t))\mathbf{u}(t) + \mathbf{E}_{i\omega}\tilde{\mathbf{w}}(t)), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \end{cases} \tag{2}$$

where $\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1, w_i(\mathbf{x}(t)) \geq 0 \forall i$, $w_i(\mathbf{x}(t))$ is the normalized grade of membership; $\mathbf{y}(t) \in \Re^l$ is the measurement output; $\mathbf{C}$ is the output matrix.

Definition 1. *[1] A system is deemed to be positive if the initial condition* $\mathbf{x}(0) = \mathbf{x}_0 \succeq 0$ *holds and the corresponding trajectory* $\mathbf{x}(t) \succeq 0$ *for all* $t \geq 0$ *is satisfied.*

Definition 2. *[1] A matrix* $\mathbf{M}$ *is called a Metzler matrix if its off-diagonal elements are non-negative:* $m_{rs} \succeq 0$, $r \neq s$.

Lemma 1. *[14, 15] System* (2) *is a positive system if* $\mathbf{A}_i(\mathbf{x}(t))$ *is a Metzler matrix,* $\mathbf{B}_i(\mathbf{x}(t)) \succeq 0$, $\mathbf{B}_{i\omega} \succeq 0$, $\mathbf{D}_i(\mathbf{x}(t)) \succeq 0$, $\mathbf{E}_i(\mathbf{x}(t)) \succeq 0$, $\mathbf{E}_{i\omega} \succeq 0$ *and* $\mathbf{C} \succeq 0$.

Under zero initial conditions, the L_1-induced performance of the positive polynomial fuzzy system (2) is defined as follows:

$$||\mathbf{z}(t)||_{L_1} < \gamma ||\tilde{\mathbf{w}}(t)||_{L_1}. \tag{3}$$

2.3 Polynomial Fuzzy Controller Design

In the following, a c-rule PFC in terms of the SOF control scheme is designed:

$$\begin{aligned} &\text{Rule } j : \text{IF } g_1(\mathbf{y}(t)) \text{ is } N_1^j \text{ AND} \ldots \text{AND } g_\Omega(\mathbf{y}(t)) \text{ is } N_\Omega^j \\ &\qquad \text{THEN } \mathbf{u}(t) = \mathbf{v}\mathbf{K}_j(\mathbf{y}(t))\mathbf{y}(t), \end{aligned} \tag{4}$$

where $g_\beta(\mathbf{y}(t))$ is the premise variable, Ω is a positive integer; N_β^j is the fuzzy set of j-th rule corresponding to the function $g_\beta(\mathbf{y}(t))$; $\mathbf{K}_j(\mathbf{y}(t)) \in \Re^{1\times l}$ is the feedback gain to be determined; $\mathbf{v} \in \Re^{m\times 1}$ is a given vector.

Taking $\mathbf{y}(t)$ into the PFC (4), we have:

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{y}(t))\mathbf{v}\mathbf{K}_j(\mathbf{y}(t))\mathbf{y}(t) = \sum_{j=1}^{c} m_j(\mathbf{y}(t))\mathbf{v}\mathbf{K}_j(\mathbf{y}(t))\mathbf{C}\mathbf{x}(t), \tag{5}$$

where $\sum_{j=1}^{c} m_j(\mathbf{y}(t)) = 1, m_j(\mathbf{y}(t)) \geq 0, \forall j$, $m_j(\mathbf{y}(t))$ is the normalized grade of membership.

Remark 1. The vector $\mathbf{v}$ is a non-zero constant vector which is given in advance by users. This method can facilitate the introduction of the nonsingular transformation technique which works effectively for solving the non-convex problem.

Remark 2. So far, some authors kept a watchful eye on unstable open-loop general systems [11, 16], which means only the positivity of the closed-loop systems require to be ensured. However, the work in [15] ensured both open-loop systems and closed-loop systems to be positive. In this paper, we focus on the latter case that the positivity of both polynomial fuzzy model and closed-loop positive polynomial fuzzy control system should be maintained.

For simplicity, $w_i(\mathbf{x}(t))$, $m_j(\mathbf{y}(t))$, $\mathbf{x}(t)$ and $\mathbf{y}(t)$ will be abbreviated as $w_i(\mathbf{x})$, $m_j(\mathbf{y})$, $\mathbf{x}$ and $\mathbf{y}$, respectively.

3 Stability and Positivity Analysis

In this section, we will give the stability and positivity analysis under the L_1-induced performance for the closed-loop positive polynomial fuzzy control system based on the Lyapunov theory and the L_1-induced performance index. Convex SOS-based conditions will be derived by employing some useful techniques to solve non-convex terms.

3.1 Closed-Loop Positive Polynomial Fuzzy Control Systems

Based on (2) and (5), we have the following closed-loop positive polynomial fuzzy control system:

$$\begin{cases} \dot{\mathbf{x}} = \sum\limits_{i=1}^{p}\sum\limits_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y})\Big(\big(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{v}\mathbf{K}_j(\mathbf{y})\mathbf{C}\big)\mathbf{x} + \mathbf{B}_{i\omega}\tilde{\mathbf{w}}\Big), \\ \mathbf{z} = \sum\limits_{i=1}^{p}\sum\limits_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y})\Big((\mathbf{D}_i(\mathbf{x}) + \mathbf{E}_i(\mathbf{x})\mathbf{v}\mathbf{K}_j(\mathbf{y})\mathbf{C})\mathbf{x} + \mathbf{E}_{i\omega}\tilde{\mathbf{w}}\Big), \\ \mathbf{y} = \mathbf{C}\mathbf{x}. \end{cases} \tag{6}$$

3.2 Stability Analysis of Closed-Loop Positive Polynomial Fuzzy Control Systems

The following Lyapunov function candidate [15] is chosen to study the stability of the closed-loop positive polynomial fuzzy control system (6):

$$V(t) = \lambda^T \mathbf{x}, \tag{7}$$

where $\lambda = [\lambda_1, \ldots, \lambda_n]^T \succ 0$ is a vector to be determined.

Then we have the time derivative of $V(t)$ as follows:

$$\begin{aligned}\dot{V}(t) &= \lambda^T \dot{\mathbf{x}} \\ &= \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(\mathbf{x}) m_j(\mathbf{y}) \lambda^T \Big(\mathbf{B}_{i\omega} \tilde{\mathbf{w}} + \big(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x}) \mathbf{v} \mathbf{K}_j(\mathbf{y}) \mathbf{C}\big) \mathbf{x} \Big).\end{aligned} \tag{8}$$

In the following, the L_1-induced performance will be taken into consideration, and the L_1-induced performance index is given:

$$J = \int_0^{\infty} ||\mathbf{z}||_{L_1} - \gamma ||\tilde{\mathbf{w}}||_{L_1} d_t, \tag{9}$$

Combining stability theory and performance index, the equality (9) can be dealt with as following:

$$\begin{aligned}J &= \int_0^{\infty} ||\mathbf{z}||_{L_1} - \gamma ||\tilde{\mathbf{w}}||_{L_1} + \dot{V} - \dot{V} d_t = \int_0^{\infty} \sum_{k=1}^{q} \mathbf{z} - \gamma \sum_{k=1}^{p} \tilde{\mathbf{w}} + \dot{V} d_t - V(\infty) \\ &= \int_0^{\infty} \mathbf{I}_1^T \mathbf{z} - \gamma \mathbf{I}_2^T \tilde{\mathbf{w}} + \dot{V} d_t - V(\infty),\end{aligned} \tag{10}$$

where $\mathbf{I}_1 \in \Re^q$ and $\mathbf{I}_2 \in \Re^p$ are vectors with all of the elements being 1.

The term $V(\infty)$ is equal to 0 when $t \to \infty$, then by taking $\mathbf{z}$ and (8) into (10), we have

$$\begin{aligned}J &= \int_0^{\infty} \mathbf{I}_1^T \mathbf{z} - \gamma \mathbf{I}_2^T \tilde{\mathbf{w}} + \dot{V} d_t \\ &= \int_0^{\infty} \mathbf{I}_1^T \Big(\sum_{i=1}^{p} \sum_{j=1}^{c} w_i(\mathbf{x}) m_j(\mathbf{y}) \big(\mathbf{E}_{i\omega} \tilde{\mathbf{w}} + (\mathbf{D}_i(\mathbf{x}) + \mathbf{E}_i(\mathbf{x}) \mathbf{v} \mathbf{K}_j(\mathbf{y}) \mathbf{C}) \mathbf{x} \big) \Big) \\ &\quad - \gamma \mathbf{I}_2^T \tilde{\mathbf{w}} + \lambda^T \Big(\sum_{i=1}^{p} \sum_{j=1}^{c} w_i(\mathbf{x}) m_j(\mathbf{y}) \big(\mathbf{B}_{i\omega} \tilde{\mathbf{w}} + (\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x}) \mathbf{v} \mathbf{K}_j(\mathbf{y}) \mathbf{C}) \mathbf{x} \big) \Big) d_t \\ &= \int_0^{\infty} \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(\mathbf{x}) m_j(\mathbf{y}) \Big(\big(\mathbf{I}_1^T \mathbf{E}_{i\omega} - \gamma \mathbf{I}_2^T + \lambda^T \mathbf{B}_{i\omega} \big) \tilde{\mathbf{w}} \\ &\quad + \big(\mathbf{I}_1^T (\mathbf{D}_i(\mathbf{x}) + \mathbf{E}_i(\mathbf{x}) \mathbf{v} \mathbf{K}_j(\mathbf{y}) \mathbf{C}) \\ &\quad + \lambda^T (\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x}) \mathbf{v} \mathbf{K}_j(\mathbf{y}) \mathbf{C}) \big) \mathbf{x} \Big) d_t.\end{aligned} \tag{11}$$

In order to facilitate the analysis, we define:

$$\mathbf{Q}_{1ij}(\mathbf{x},\mathbf{y}) = \mathbf{I}_1^T\mathbf{E}_{i\omega} - \gamma\mathbf{I}_2^T + \lambda^T\mathbf{B}_{i\omega}, \tag{12}$$
$$\mathbf{Q}_{2ij}(\mathbf{x},\mathbf{y}) = \mathbf{I}_1^T\big(\mathbf{D}_i(\mathbf{x}) + \mathbf{E}_i(\mathbf{x})\mathbf{v}\mathbf{K}_j(\mathbf{y})\mathbf{C}\big) + \lambda^T\big(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{v}\mathbf{K}_j(\mathbf{y})\mathbf{C}\big). \tag{13}$$

Based on (11), the inequality $J < 0$ holds if $\mathbf{Q}_{1ij}(\mathbf{x},\mathbf{y}) \prec 0$ and $\mathbf{Q}_{2ij}(\mathbf{x},\mathbf{y}) \prec 0$ for all i and j are satisfied. However, we find that the term $\lambda^T\mathbf{B}_i(\mathbf{x})\mathbf{v}\mathbf{K}_j(\mathbf{y})\mathbf{C}$ is non-convex. To solve the non-convex problem, we assume that $\mathbf{B}_i(\mathbf{x}) = \mathbf{B}$ for all i. Inspired by the nonsingular transformation technique in [17], we have:

$$\bar{\mathbf{B}} = \mathbf{B}\mathbf{v}, \mathbf{\Gamma} = \begin{bmatrix} (\bar{\mathbf{B}}^T\bar{\mathbf{B}})\bar{\mathbf{B}}^T \\ \mathrm{ortc}(\bar{\mathbf{B}}) \end{bmatrix}, \tag{14}$$

where $\mathrm{ortc}(\bar{\mathbf{B}}) \in \Re^{(n-1)\times n}$ represents the orthogonal complement of $\bar{\mathbf{B}}$ which satisfies the following equality:

$$\mathbf{\Gamma}\bar{\mathbf{B}} = [1 \quad ; \quad \mathbf{0}_{n-1}] \tag{15}$$

where $\mathbf{0}_{n-1}$ means that the elements in $\mathbf{0}_{n-1}$ are all zero.

Therefore, the non-convex term $\lambda^T\mathbf{B}_i(\mathbf{x})\mathbf{v}\mathbf{K}_j(\mathbf{y})\mathbf{C}$ can be dealt with further:

$$\lambda^T\mathbf{B}_i(\mathbf{x})\mathbf{v}\mathbf{K}_j(\mathbf{y})\mathbf{C} = \lambda^T\mathbf{\Gamma}^{-1}\mathbf{\Gamma}\mathbf{B}\mathbf{v}\mathbf{K}_j(\mathbf{y})\mathbf{C} = p_1\mathbf{K}_j(\mathbf{y})\mathbf{C} = \mathbf{Z}_j(\mathbf{y})\mathbf{C}, \tag{16}$$

where $\lambda^T\mathbf{\Gamma}^{-1} = \mathbf{p}^T$ and $p_1\mathbf{K}_j(\mathbf{y}) = \mathbf{Z}_j(\mathbf{y}) \in \Re^{1\times l}$; as the first element in $\mathbf{p}^T$, p_1 is a positive value and the detailed proof can refer to [4].

Inducing (16) and $\lambda^T = \mathbf{p}^T\mathbf{\Gamma}$ into (13), we can obtain

$$\mathbf{I}_1^T\mathbf{D}_i(\mathbf{x}) + \mathbf{I}_1^T\mathbf{E}_i(\mathbf{x})\mathbf{v}\mathbf{Z}_j(\mathbf{y})/p_1\mathbf{C} + \mathbf{p}^T\mathbf{\Gamma}\mathbf{A}_i(\mathbf{x}) + \mathbf{Z}_j(\mathbf{y})\mathbf{C} \prec 0. \tag{17}$$

It can be seen that there still is a non-convex term in (17). In this case, we can get around this obstacle by dividing the inequality (17) into two parts. Then the inequality (17) can be ensured if the following two inequalities are satisfied:

$$\mathbf{I}_1^T\mathbf{E}_i(\mathbf{x})\mathbf{v}\mathbf{Z}_j(\mathbf{y})/p_1\mathbf{C} \prec 0, \tag{18}$$
$$\mathbf{I}_1^T\mathbf{D}_i(\mathbf{x}) + \mathbf{p}^T\mathbf{\Gamma}\mathbf{A}_i(\mathbf{x}) + \mathbf{Z}_j(\mathbf{y})\mathbf{C} \prec 0. \tag{19}$$

Due to $p_1 > 0$, so we can multiply both sides of (18) by p_1, then we have

$$\mathbf{I}_1^T\mathbf{E}_i(\mathbf{x})\mathbf{v}\mathbf{Z}_j(\mathbf{y})\mathbf{C} \prec 0. \tag{20}$$

Based on the above analysis, the convex stability conditions can be derived. However, not only should the stability conditions be guaranteed, but also the positivity conditions need to be satisfied. Hence, in the following, we will analyze the positivity conditions. From the closed-loop positive polynomial fuzzy systems (6), we can get the positivity conditions as follows:

$$\begin{aligned} &\mathbf{A}_i(\mathbf{x}) + \mathbf{B}\mathbf{v}\mathbf{K}_j(\mathbf{y})\mathbf{C} \quad \text{is a Metzler}, \\ &\mathbf{D}_i(\mathbf{x}) + \mathbf{E}_i(\mathbf{x})\mathbf{v}\mathbf{K}_j(\mathbf{y})\mathbf{C} \succeq 0. \end{aligned} \tag{21}$$

Recall the definition 2 and take $\mathbf{K}_j(\mathbf{y}) = \mathbf{Z}_j(\mathbf{y})/p_1$ into (21), we can obtain

$$\begin{aligned} p_1 a_{irs}(\mathbf{x}) + \mathbf{b}_r \mathbf{v} \mathbf{Z}_j(\mathbf{y})\mathbf{c}_s \succeq 0, \forall r \neq s \\ p_1 \mathbf{D}_i(\mathbf{x}) + \mathbf{E}_i(\mathbf{x})\mathbf{v}\mathbf{Z}_j(\mathbf{y})\mathbf{C} \succeq 0, \end{aligned} \tag{22}$$

where $a_{irs}(\mathbf{x})$ is in the r-th row and s-th column of $\mathbf{A}_i(\mathbf{x})$, $\mathbf{b}_r$ is the r-th row of $\mathbf{B}$ and $\mathbf{c}_s$ is the s-th column of $\mathbf{C}$.

Based on the analysis above, we can summarize the results in Theorem 1.

Theorem 1. *The positive polynomial fuzzy model* (2) *can be controlled to be asymptotically stable and positive by the SOF PFC* (6) *under L_1-induced performance if there exist vectors $\mathbf{Z}_j(\mathbf{y}) \in \Re^{1\times l}$, $\mathbf{p} \in \Re^n$ such that the following SOS-based positivity and stability conditions are satisfied:*

$$p_1 a_{irs}(\mathbf{x}) + \mathbf{b}_r \mathbf{v} \mathbf{Z}_j(\mathbf{y})\mathbf{c}_s \ \textit{is SOS} \ \forall\, r \neq s, i, j; \tag{23}$$

$$\upsilon^T \Big(diag\big(p_1 \mathbf{D}_i(\mathbf{x}) + \mathbf{E}_i(\mathbf{x})\mathbf{v}\mathbf{Z}_j(\mathbf{y})\mathbf{C}\big)\Big)\upsilon \ \textit{is SOS}; \tag{24}$$

$$\upsilon^T \Big(diag\big(\mathbf{p}^T \mathbf{\Gamma} - \epsilon_1 \mathbf{I}\big)\Big)\upsilon \ \textit{is SOS}; \tag{25}$$

$$-\upsilon^T \Big(diag\big(\mathbf{I}_1^T \mathbf{E}_i(\mathbf{x})\mathbf{v}\mathbf{Z}_j(\mathbf{y})\mathbf{C} + \epsilon_2(\mathbf{x})\mathbf{I}\big)\Big)\upsilon \ \textit{is SOS} \ \forall\, i, j; \tag{26}$$

$$-\upsilon^T \Big(diag\big(\mathbf{I}_1^T \mathbf{D}_i(\mathbf{x}) + \mathbf{p}^T \mathbf{\Gamma} \mathbf{A}_i(\mathbf{x}) + \mathbf{Z}_j(\mathbf{y})\mathbf{C} + \epsilon_3(\mathbf{x})\mathbf{I}\big)\Big)\upsilon \ \textit{is SOS} \ \forall\, i, j; \tag{27}$$

$$-\upsilon^T \Big(diag\big(\mathbf{I}_1^T \mathbf{E}_{i\omega} - \gamma \mathbf{I}_2^T + \mathbf{p}^T \mathbf{\Gamma} \mathbf{B}_{i\omega} + \epsilon_4 \mathbf{I}\big)\Big)\upsilon \ \textit{is SOS} \ \forall\, i, j; \tag{28}$$

where $\upsilon \in \Re^n$ is an arbitrary vector independent of $\mathbf{x}$ and $\mathbf{y}$; $\epsilon_1 > 0$ and $\epsilon_4 > 0$ are predefined scalars and $\epsilon_2(\mathbf{x}) > 0$ and $\epsilon_3(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$ are predefined scalar polynomials. The feedback gain is $\mathbf{K}_j(\mathbf{y}) = \mathbf{Z}_j(\mathbf{y})/p_1$.

Remark 3. The conditions (23) and (24) are the positivity conditions which are used to guarantee the positivity of the closed-loop positive polynomial fuzzy systems. The conditions (25), (26), (27) and (28) are employed to ensure the stability under the L_1-induced performance.

4 Simulation Example

In this section, a simulation example is given to validate the theory in this paper.

4.1 Scenario

The 3-rule positive polynomial fuzzy model is shown as follows:

$$\mathbf{x} = [\, x_1 \ x_2 \,]^T, \mathbf{v} = \begin{bmatrix} 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 \ 0 \end{bmatrix}, \mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\mathbf{A}_1(x_1) = \begin{bmatrix} 0.11 & 1 \\ 0.55 + 0.5a & -0.85 - 0.12\mathbf{x}_1^2 + 0.11\mathbf{x}_1 \end{bmatrix},$$

$$\mathbf{A}_2(x_1) = \begin{bmatrix} 0.14 & 1.2 \\ 0.72 + 0.5a & -1.37 - 0.24\mathbf{x}_1^2 + 0.25\mathbf{x}_1 \end{bmatrix},$$

$$\mathbf{A}_3(x_1) = \begin{bmatrix} 0.19 & 1.5 \\ 0.85 + 0.5a & -1.9 - 0.42\mathbf{x}_1^2 + 0.31\mathbf{x}_1 \end{bmatrix},$$

$$\mathbf{D}_1(x_1) = \begin{bmatrix} 0.45 + 0.5b & 1.52 + 0.12\mathbf{x}_1^2 + 0.31\mathbf{x}_1 \\ 1.6 & 0.38 \end{bmatrix},$$

$$\mathbf{D}_2(x_1) = \begin{bmatrix} 0.63 + 0.5b & 1.09 + 0.15\mathbf{x}_1^2 + 0.24\mathbf{x}_1 \\ 1.1 & 0.22 \end{bmatrix},$$

$$\mathbf{D}_3(x_1) = \begin{bmatrix} 0.89 + 0.5b & 2.11 + 0.18\mathbf{x}_1^2 + 0.11\mathbf{x}_1 \\ 2.3 & 0.65 \end{bmatrix},$$

$$\mathbf{B}_{1\omega} = \begin{bmatrix} 1.4 \\ 0.16 \end{bmatrix}, \mathbf{B}_{2\omega} = \begin{bmatrix} 1.5 \\ 0.35 \end{bmatrix}, \mathbf{B}_{3\omega}(x_1) = \begin{bmatrix} 1.8 \\ 0.44 \end{bmatrix},$$

$$\mathbf{E}_1 = \begin{bmatrix} 1.41 \\ 0.46 \end{bmatrix}, \mathbf{E}_2 = \begin{bmatrix} 1.65 \\ 0.15 \end{bmatrix}, \mathbf{E}_3 = \begin{bmatrix} 1.98 \\ 0.04 \end{bmatrix},$$

$$\mathbf{E}_{1\omega} = \begin{bmatrix} 1.54 \\ 1.06 \end{bmatrix}, \mathbf{E}_{2\omega} = \begin{bmatrix} 2.25 \\ 0.55 \end{bmatrix}, \mathbf{E}_{3\omega} = \begin{bmatrix} 0.18 \\ 0.84 \end{bmatrix},$$

where a and b are constant scalars.

The open-loop system is a positive system because the off-diagonal elements in $\mathbf{A}_i(x_1), i \in \{1,2,3\}$ are non-negative; the elements of $\mathbf{D}_i(x_1), i \in \{1,2,3\}, \mathbf{B}_{i\omega}, \mathbf{E}_i, \mathbf{E}_{i\omega}, \mathbf{C}$ and $\mathbf{B}$ are non-negative. In addition, $\mathbf{v}$ is a given vector and chosen arbitrarily in the simulation satisfying $\mathbf{B}\mathbf{v} = \bar{\mathbf{B}} = [1; \quad 1] \succeq 0$. The disturbance signal is $\tilde{\mathbf{w}}(t) = 4.5e^{-t}|cos(2t)|$.

A 2-rule PFC is designed. The MFs of the model and the controller are same as the ones in the simulation section in [4], which are illustrated in Fig. 1. Theorem 1 in this paper is verified with $1 \leq a \leq 10$ at the interval of 1 and $2 \leq b \leq 12$ at the interval of 1. And the scalars ϵ_1, $\epsilon_2(\mathbf{x})$, $\epsilon_3(\mathbf{x})$ and ϵ_4 are set as 0.001, the highest degree of $\mathbf{Z}_j(x_1)$ is 2.

Table 1. the Feedback Gains $K_j(x_1)$, λ and γ Obtained by Theorem 1.

(a,b)	$K_j(x_1)$	λ	γ
$a=1$	$K_1(x_1) = 1.6444e^{-17}x_1^2 - 0.9545$	$p_1 = 42.8049$	44.6147
$b=2$	$K_2(x_1) = -1.3681e^{-17}x_1^2 - 0.9545$	$p_2 = 4.5212$	
$a=6$	$K_1(x_1) = -1.19703e^{-15}x_1^2 - 2.2172$	$p_1 = 505.1437$	532.2583
$b=7$	$K_2(x_1) = 4.7696e^{-15}x_1^2 - 2.2172$	$p_2 = 35.8998$	
$a=7$	$K_1(x_1) = 2.3482e^{-12}x_1^2 - 2.7222$	$p_1 = 49.2535$	51.4175
$b=9$	$K_2(x_1) = -3.8775e^{-12}x_1^2 - 2.7222$	$p_2 = 4.9574$	
$a=10$	$K_1(x_1) = -1.1706e^{-12}x_1^2 - 3.4783$	$p_1 = 83.7958$	87.8538
$b=12$	$K_2(x_1) = -1.8539e^{-12}x_1^2 - 3.4783$	$p_2 = 7.2983$	

4.2 Feasibility Analysis

From the Fig. 2, it makes clear that the SOF PFC has the ability to achieve the stability and positivity under the L_1-induced performance for an open-loop

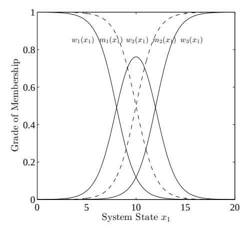

Fig. 1. Membership functions of positive polynomial fuzzy model and SOF PFC.

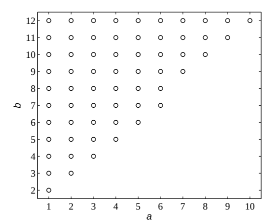

Fig. 2. Stability region given by Theorem 1

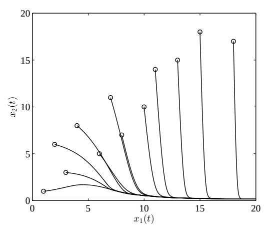

Fig. 3. Phase plot of the states x_1 and x_2 for $a = 1, b = 2$ for the open-loop system.

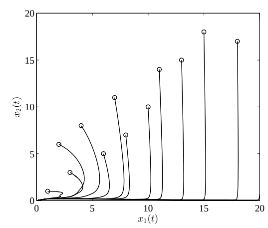

Fig. 4. Phase plot of the states x_1 and x_2 for $a = 1, b = 2$ for the closed-loop system.

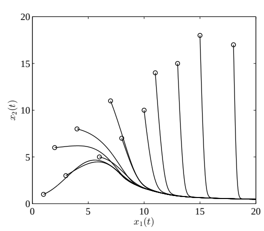

Fig. 5. Phase plot of the states x_1 and x_2 for $a = 6, b = 7$ for the open-loop system.

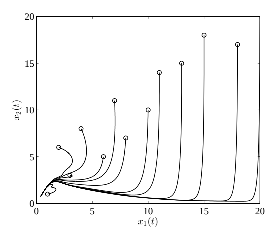

Fig. 6. Phase plot of the states x_1 and x_2 for $a = 6, b = 7$ for the closed-loop system.

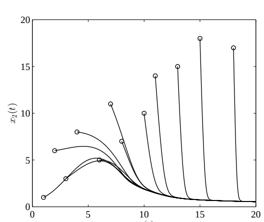

Fig. 7. Phase plot of the states x_1 and x_2 for $a = 7, b = 9$ for the open-loop system.

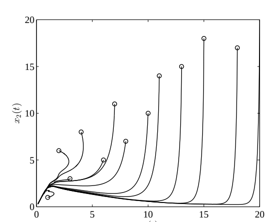

Fig. 8. Phase plot of the states x_1 and x_2 for $a = 7, b = 9$ for the closed-loop system.

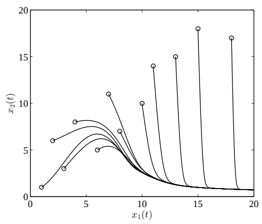

Fig. 9. Phase plot of the states x_1 and x_2 for $a = 10, b = 12$ for the open-loop system.

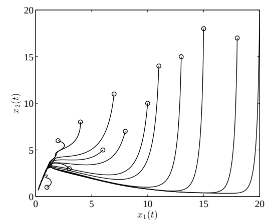

Fig. 10. Phase plot of the states x_1 and x_2 for $a = 10, b = 12$ for the closed-loop system.

unstable positive polynomial fuzzy system. Meanwhile, to further confirm the correctness of this method, we pick out several feasible points (a, b) to check their phase plots of x_1 and x_2 obeying various initial conditions. In Fig. 2, we choose (1, 2), (6, 7), (7, 9) and (10, 12), respectively. λ , γ and $\mathbf{K}_j(x_1)$ corresponding to these feasible points (a, b) are checked as well and appear in Table I . The phase plots of these points can refer to Figs. 3 to 10, where Figs. 3, 5, 7 and 9 are the phase plots of these points for open-loop systems, and Figs. 4, 6, 8 and 10 are the phase plots of these points for closed-loop systems. From Figs. 3 to 10, we can see that the open-loop system is positive but unstable, while the closed-loop system is asymptotically stable and positive.

According to the phase plots, the positive polynomial fuzzy system can be driven to the origin. Therefore, it is clear that the unstable open-loop positive polynomial fuzzy systems can be controlled to be stable and positive under L_1-induced performance by the SOF PFC.

5 Conclusion

In this paper, under L_1-induced performance, the SOF control synthesis and stability analysis for closed-loop positive polynomial fuzzy control systems have been studied. The non-convex terms in the stability conditions have been transformed into convex ones so that the MATLAB third-party toolbox SOSTOOL can be used to obtain the feasible solution. The SOF strategy has been introduced in the PFC design so that the unstable open-loop positive systems can be controlled to be asymptotically stable and positive. Meanwhile, taking the L_1-induced performance into consideration, the SOS-based stability and positivity conditions have been obtained in terms of the Lyapunov stability theory. An example has been given to demonstrate the correctness of the theorem, in addition, the stability region and the phase plots of some feasible points have been shown in this paper.

References

1. M. A. Rami and F. Tadeo, "Controller synthesis for positive linear systems with bounded controls," *IEEE Trans. Circuits Syst.*, vol. 54, no. 2, pp. 151–155 (2007).
2. Bhattacharyya and S. Patra, "Static Output-Feedback Stabilization for MIMO LTI Positive Systems using LMI-based Iterative Algorithms," *IEEE Control Syst. Letters*, vol. 2, no. 2, pp. 242–247 (2018).
3. Y. Ju and X. Zhu and Y. Sun, "Stability Analysis of Continuous-Time Positive Switched Linear Systems," in *Prof. 18th Inter. Conf. Control, Autom. and Syst.*, pp. 1062–1065 (2018).
4. A. Meng, H. K. Lam, Y. Yu, X. Li, and F. Liu, "Static output feedback stabilization of positive polynomial fuzzy systems," *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 3, pp. 1600–1612 (2018).
5. H. K. Lam, "A review on stability analysis of continuous-time fuzzy-model-based control systems: From membership-function-independent to membership-function-dependent analysis," *Engineer. Appli. Artif. Intel.*, vol. 67, pp. 390 – 408 (2018).
6. S. Tsai and C. Jen, "H_∞ stabilization for polynomial fuzzy time-delay system: A sum-of-squares approach," *IEEE Trans. Fuzzy Syst*, vol. 26, no. 6, pp. 3630–3644 (2018).
7. H. K. Lam, *Polynomial Fuzzy Model-Based Control Systems: Stability Analysis and Control Synthesis Using Membership Function Dependent Techniques.* Switzerland: Springer International Publishing (2016).
8. T. R. V. Steentjes and A. I. Doban and M. Lazar, "Feedback stabilization of positive nonlinear systems with applications to biological systems," in *Prof. of 2018 Euro. Control Conf.*, pp. 1619-1624 (2018).
9. S. Zhu and F. Pan and J. Feng, "Fuzzy filtering design for positive T-S fuzzy systems with Markov jumping parameters," in *Prof. of 2018 Austra. New Zealand Control Conf.*, pp. 1–4 (2018).
10. X. Li, C. Liu, H. K. Lam, F. Liu, and X. Zhao, "Stability analysis of fuzzy polynomial positive systems with time delay," in *Proc. of Int. Conf. Fuzzy Theory and Its Appl.*, pp. 24–28 (2014).
11. X. Li, H. K. Lam, F. Liu, and X. Zhao, "Stability and stabilization analysis of positive polynomial fuzzy systems with time delay considering piecewise membership functions," *IEEE Trans. Fuzzy Syst*, vol. 25, no. 4, pp. 958–971 (2017).
12. S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "Introducing SOSTOOLS: A general purpose sum of squares programming solver," in *Proc. of 41st IEEE Conf. Decision Control.*, vol. 1, pp. 741–746 (2002).
13. S. Prajna, A. Papachristodoulou, and F. Wu, "Nonlinear control synthesis by sum of squares optimization: A Lyapunov-based approach," in *Proc. of 5th Asian Control Conf.*, vol. 1, pp. 157–165 (2004).
14. L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications.* New York: John Wiley and Sons Inc. (2000).
15. J. Zhang, Z. Han, F. Zhu, and J. Huang, "Brief paper: Feedback control for switched positive linear systems," *IET Control Theory Appl.*, vol. 7, no. 3, pp. 464–469 (2013).
16. M. Min, Z. Shuqian, and Z. Chenghui, "Static output feedback control for positive systems via LP approach," in *Prof. of 31st Chin. Control Conf.*, pp. 1435–1440 (2012).
17. H. K. Lam, "Stabilization of nonlinear systems using sampled-data output-feedback fuzzy controller based on polynomial-fuzzy-model-based control approach," *IEEE Trans. Syst. Man Cybern. B. Cybern.*, vol. 42, no. 1, pp. 258–267 (2012).